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REVIEW ARTICLE

Fractional statistics: α to β

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Abstract. We review the problem of fractional statistics as it applies to two current areas of interest in condensed-matter physics: the fractional quantum Hall effect (FQHE), and high-temperature superconductors (HTSC). In the case of the former, we emphasize Haldane's recent definition of a *fractional exclusion principle*, and show a relation between this idea and the standard definition of fractional statistics in terms of a complex exchange phase. We show that a fractional exclusion principle is both appropriate and useful for the quasiparticles in the FQHE. In the case of the HTSC (where Haldane's novel definition has not been pursued), we review the experimental status of the 'anyon superconductivity' model for the HTSC. Here we find much less support for the hypothesis that the excitations are anyons. We also argue that the past neglect of Haldane's fractional exclusion principle makes the resulting theory inconsistent.

1. Introduction

What is an anyon? Why even consider the idea? The answers to these questions have evolved over the last decade and a half, thanks to a large body of work by a number of investigators from a variety of backgrounds.

It may be useful to answer the second question first. There are two reasons—as there are for most problems in physics—to think seriously about anyons: there is the aesthetic appeal of the ideas connected with fractional statistics, and there is the possibility that these ideas can help us understand physical phenomena. The anyon problem, nevertheless, has a somewhat unique fingerprint. It involves very simple and fundamental ideas, whose consequences are subtle and not easily guessed. Hence physicists from many specialties are attracted to the problem. However, so far, the restriction of these ideas to two spatial dimensions has almost completely limited their application to condensed-matter physics.

We are drawn to the problem for both reasons; however, believing that the beauty of the ideas speaks for itself better than we could elaborate it, and having the most interest in those ideas with potential for application, we propose to concentrate on the problems of fractional statistics in the context of condensed-matter physics—in particular, those two condensed-matter systems in which anyons have been seriously considered as candidates for well defined excitations: the fractional quantum Hall effect (FQHE), and high-temperature superconductors (HTSC).

The concepts of fractional statistics may also be considered to fall into two areas, namely, the by-now well known *fractional exchange parameter* α (defined by writing the exchange

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phase† as $\exp(i\alpha\pi)$), and the less well known, but equally novel, *fractional exclusion principle* (first defined by Haldane [1] in 1991, defined by us below, and symbolized by the 'exclusion coefficient' β henceforth). In fact, following Haldane's invention of the latter idea, and our own efforts [2–4] to test and apply it in the FQHE, we claim that there are now two, in principle distinct, kinds of anyons—which we will call, not surprisingly, α-anyons and β-anyons.

In discussing the FQHE we will concentrate on β, in order to emphasize the most recent work, and to avoid excessive redundancy: the significance of fractional α for this problem was pointed out in 1984 by Halperin [5] and by Arovas *et al* [6], and has been amply reviewed elsewhere, for example in [8–10].

In the case of HTSC, only the idea of fractional α has (to our knowledge) been applied. Here we will emphasize the experimental tests of this idea, and their interpretation. Our choice is again guided by an effort to avoid redundancy. There are, in fact, more reviews of 'anyon superconductivity' than we can keep track of; we mention [7, 10] and two original articles [11, 12] for entrance to the literature.

2. Fractional statistics: α and β

Let us briefly reintroduce α. We need to define an *exchange phase* which is in general distinct from a *permutation eigenvalue* of a wavefunction. This phase may be thought of as a unitary weight which is associated with paths involving exchange in a path integral [14–16], or more simply as the phase $\eta \equiv \exp(i\theta)$ which results when two identical particles are exchanged. Then α is defined by putting $\theta = \alpha\pi$. Clearly α is only defined mod 2.

If α is not an integer then η is complex, which in turn means that an exchange may be distinguished from its time-reverse. This further implies that a clockwise exchange is distinct from a counterclockwise exchange—which makes no sense unless we confine ourselves to two spatial dimensions. (Also, we clearly must allow for broken time-reversal ($\mathcal{T}$) symmetry, and for broken two-dimensional parity ($\mathcal{P}_2$).) So we very quickly obtain the 'standard' definition of fractional statistics—one which is only applicable in two dimensions, and which necessarily involves broken symmetry.

Why is it called 'fractional statistics' (rather than 'fractional multiple of π in the exchange phase')? An obvious question: 'statistics' refers to state counting in statistical mechanics; and everyone knows that the counting rules for bosons ($\eta = +1$) differ from those for fermions ($\eta = -1$). Yet in the case of fractional α, defined as above, the second phrase, although horribly unwieldy, is accurate, while the first (fractional statistics) is misleading: it suggests that we know the counting rules for objects with fractional α.

In fact, we do not. Given α, we cannot, in general, specify how states are 'filled' by identical α-anyons without further investigation. The simple answers, which we know already, are those for ideal bosons and fermions. We can formalize this by defining β to be the 'number of single-particle states' which are 'filled' (removed from the one-particle Hilbert space) by the addition of a single particle. Then ideal bosons and fermions have $\beta = 0$ and 1, respectively; and, for the simple boson and fermion cases, we get $\beta = \alpha$ (mod 2).

We use quotation marks heavily above because it has been known for some time [12, 16] that one-particle states do not provide a useful basis for the many-anyon problem—hence the obvious extension to fractional β does not work (and makes no sense: how to fill $\frac{1}{2}$ of

† Here and elsewhere we freely use the term *phase* to refer either to the complex exponential $e^{i\theta}$ or to its argument θ.

a state?).

Haldane [1] nonetheless provided a usable, non-obvious extension of these ideas in 1991. He considered the many-body wavefunctions of n identical particles. Holding the coordinates of $n-1$ particles fixed, he supposed that the many-body wavefunction could be expanded in a basis of single-particle states of the nth particle, assumed to span a single-particle Hilbert space of dimension d_n. It is crucial to Haldane's approach that d_n be finite, and, in general, depend on the number of particles present. Hence the following definition applies to elementary excitations (quasiparticles) in a condensed-matter system. Haldane's exclusion coefficient β is defined by the change in d_n when the number of particles changes:

$$\beta = -(\Delta d_n / \Delta n)\,. \tag{1}$$

(For rational β, Δn can be chosen to make Δd_n an integer.) It is obviously not always simple, or even possible, to extract a single-particle dimension d_n in this way. An approach which is often more convenient is based on the dimension D_n of the many-particle Hilbert space. D_n and d_n are related by [1–4]

$$D_n = \binom{d_n + n - 1}{n} \tag{2}$$

where $\binom{n}{k} \equiv n!/(n-k)!k!$ is the usual 'choose' notation. The definition (1) is equivalent to

$$d_n = d_1 - \beta(n-1) \tag{3}$$

giving finally

$$D_n = \binom{d_1 + (1-\beta)(n-1)}{n}\,. \tag{4}$$

(Notice that it is essential to hold the Hamiltonian *and* the boundary conditions, including the area, constant as the number n is varied, in order for a computation of β to make sense [1].)

The above then gives us a second 'statistics parameter' (β), and therefore a second meaning for the term 'fractional statistics' (i.e. fractional β). The definitions (1)–(4) obviously make sense in the integral cases, i.e. ideal bosons and fermions. Haldane [1] offered two non-trivial examples involving fractional β: spinons in a disordered 'RVB' spin liquid [17], and charged excitations in the FQHE. In subsequent work [2–4] we tested Haldane's argument for the FQHE using an exact numerical computation of β, via D_n. We are not aware of any other work which directly addresses the question of fractional β.

Hence our knowledge of β-anyons is quite limited at present. We believe, however, that the idea is as important as that of α-anyons. A basis for this statement might be found in a summary of the kinds of evidence we find for both α- and β-anyons, as follows.

FQHE. The FQHE is the only real, physical system for which there is strong (albeit entirely indirect, and almost entirely theoretical) evidence for fractional statistics. The evidence for fractional α is mostly variational (cf [6, 18]), but nevertheless convincing and widely accepted. We have shown [2–4] that β can be calculated independently of any variational ansatz, and that it is fractional. We have further shown that α and β are closely linked for 2D particles in the lowest Landau level (LLL)—which, to a good approximation, describes the FQHE.

Antiferromagnets and HTSC. Disordered antiferromagnets ('spin liquids' having only short-range antiferromagnetic order—including doped ones such as, possibly, the HTSC) have been thought to support excitations which are α-anyons [11, 19, 20]. It appears to be difficult to produce a spin liquid from any reasonable Hamiltonian†. However, given such a state supporting spinon excitations, Haldane has offered a simple and convincing argument, along with an exact solution for a 1D problem, that $\beta = \frac{1}{2}$ for the spinons [1]. The arguments [11, 19, 20] that $\alpha = \frac{1}{2}$ for the spinons as well are more tenuous. Finally, experiments (to be reviewed in detail below) seeking signs of α-anyons (in terms of broken time-reversal symmetry) in the HTSC have failed to provide convincing evidence for the idea.

Others. As already hinted in the previous paragraph, the notion of fractional β is not restricted to two spatial dimensions. It is therefore clear that fractional β need not always be accompanied by fractional α. The converse, however, may not be true: it may be that, in nature‡, fractional α is inevitably accompanied by fractional β. Certainly, if the complex exchange phase arises from vortices in an incompressible fluid [15] (as is the case for the quasiparticles in the FQHE), then [1, 21] a fractional β follows. (We will make this connection clearer in the next section.) Hence fractional β-statistics may in fact be the more general phenomenon of the two.

We hope to have answered the question 'what are anyons?' in this section. The answer is longer, and more interesting, than it used to be. In the next two sections we address the question 'why consider them?' in the more restricted and pragmatic sense, i.e. 'where are they?'. We will mostly *not* address the most pragmatic question (for physicists), namely, 'what testable predictions come from anyon models?'. This is primarily because the answer is 'very few'. (In fact, one might argue that the primitive state of anyon *theory* is evidenced by the fact the we have only just realized how their Hilbert space behaves with filling, and have not yet systematically explored the consequences.)

3. Fractional quantum Hall effect

3.1. Preliminary: α and β in the FQHE

Why do we believe that quasiparticles in the FQHE are α-anyons? The argument, at least for QH, is rather simple. (From now on we use 'QH' and 'QE' to mean quasihole(s) and quasielectron(s), respectively, and retain the term quasiparticles to mean either.) A QH arises [18] as an extra zero in the wavefunction of all the electrons. This zero represents an extra unit of (canonical) angular momentum for each electron, due to an extra flux quantum in the area (relative to the $\nu = 1/m$ state, where ν is the Landau-level filling fraction). Hence the QH is a vortex—which gives it one unit of *fictitious* flux even though the true magnetic flux is not bound to it. Since the phase of the electrons winds around each vortex by 2π (just as the phase of a unit charge winds by 2π when it is transported around a flux quantum), the electron number of the QH is its fictitious 'charge' (also its *true* charge since the electrons are charged). Thus, QH represent a composite of fictitious charge bound to fictitious flux. This 'charge'/'flux' language is useful since it allows us to obtain the *non*-fictitious exchange phase as an 'Aharonov–Bohm' phase [16]. In convenient units, the exchange parameter α is then just 'charge'$\times$'flux'—which is $1/m$ for the QH [6].

† A nice overview of spin liquids may be found in the appendix of [8].

‡ One can of course define *models* for which α is fractional and β is an integer. On this point, see the discussion in section 5.

The QE then represents an 'anti-zero' (created by the electrons to absorb a *deficit* of one flux quantum, again relative to $\nu = 1/m$) which is less well understood in detail [18]. Nevertheless it is clear that both the vorticity and the charge of the QE are just the negative of those for a QH—giving† $\alpha_{\mathrm{QE}} = \alpha_{\mathrm{QH}}$.

With this picture, and with one more ingredient, we can rather readily understand Haldane's argument for $\beta_{\mathrm{QE/QH}}$ in a simple way. The extra ingredient is the single-particle Hilbert space dimension for the quasiparticles, which is obtained as follows. We consider the quasiparticles as the relevant degrees of freedom, and note that [6, 21, 22] to the quasiparticles, the electrons appear as zeroes, i.e. as 'flux'. Hence the quasiparticles may be viewed as quantum mechanical 'charges' in their lowest Landau level, and the dimension of their (single-particle) Hilbert space is just the total 'flux' they see—which is N_e, the number of electrons.

We now add quasiparticles (let us say, to be specific, QH) *at fixed boundary conditions*—which here means fixed true flux. Hence we can only change the QH number n by changing the electron number N_e—which changes the Hilbert space dimension. We then get

$$\beta_{\mathrm{QH}} = -\Delta d_n/\Delta n = -\Delta N_e/\Delta n = +1/m\,. \tag{5}$$

Adding QH is accomplished by removing electrons, thus reducing the Hilbert space for further QH.

Clearly this argument amounts to equating $-\beta$ with the charge of the QH (in units of the electronic charge). Assuming our model for QE is correct, the same statement should hold for them; this gives

$$\beta_{\mathrm{QE}} = -1/m\,. \tag{6}$$

3.2. α, β, and angular momentum in the lowest Landau level

The above arguments, leading to $\beta_{\mathrm{QH/QE}} = \pm 1/m$, are quite simple and apparently robust, since they rely only on the unquestioned charge and vorticity of the quasiparticles—plus the assumption that we know how to count the Hilbert space dimension for vortices. In order to accept these β values we need, however, to augment the above arguments with others, and/or with specific calculations. We will do both here. First we offer another argument, using pseudo-wavefunctions for the quasiparticles.

Consider n α-anyons confined to a disk of radius R, in a uniform magnetic field (R and subsequent lengths are in units of the magnetic length). To be definite, suppose that positive-charge fermions (with attached 'upwards' flux to make them α-anyons) move in the xy plane, and that the magnetic field points down. Treating the external field in the symmetric gauge, and the attached flux in the singular gauge, it can be shown [23] that n-anyon wavefunctions

$$\Psi = \prod_{i<j=1}^{n} (z_i - z_j)^{\alpha}\,\Phi \tag{7}$$

$$\Phi = f(z_1, z_2, \ldots, z_n)\mathrm{e}^{-\sum_k |z_k|^2/4} \tag{8}$$

where f is any symmetric polynomial, form a set of degenerate ground states (for ideal particles). These can be thought of as lying in a lowest Landau level. Our task is to count the number of such states, to extract an exclusion coefficient β using (4). We can view

† Note that this sign convention assumes that there is an 'absolute clockwise' in the plane which is the same for both α_{QE} and α_{QH}. Obviously the absolute sign of both of these αs depends on this convention; they have the same sign if they share the same sense of clockwise.

Φ as a wavefunction for n bosons in the lowest Landau level. (These are not simply the anyons with the flux tubes removed—the probability density given by $|\Phi|^2$ is not simply proportional to that given by $|\Psi|^2$.) Our goal is to find the number of linearly independent fs which keep the particles within the finite area. The most compact wavefunction of the form (8) has $f = 1$. For this f, Φ has all of the bosons in the single-particle state with angular momentum index $k = 0$. The bosons are all near $z = 0$, but the Jastrow factor spreads out the anyons described by Ψ. Now place one particle at the edge. That is, put $f = f_M = z_1^M + z_2^M + \cdots + z_n^M$, where M (to be found shortly) is the largest value for which all the anyons are within the finite area. With this f, $n - 1$ bosons are near the centre, and one near the edge. Now consider the first term in the symmetric sum f_M, which places z_1 near the edge and the other z_i near the origin. Then in Ψ it is easy to see that the maximum power to which z_1 is raised is $z_1^{M+\alpha(n-1)}$. This gives a peak in $|\Psi|^2$ at $|z_1| \sim M + \alpha(n-1)$. For this to be within the prescribed area, the maximum M is $M = R - \alpha(n-1)$. In general, when the bosons are placed in any of the single-particle states $k = 0$ to M, the anyons stay within the finite area. The number of such single-boson states is $M + 1$, and the number of n-boson states is

$$\binom{M+1+n-1}{n} = \binom{R+1+(1-\alpha)(n-1)}{n}. \tag{9}$$

Each of these states corresponds to an anyonic Ψ. Thus we have found that the number of n-anyon states is of the form (4), with

$$\beta = \alpha\,. \tag{10}$$

We know immediately, however, that this result is not entirely correct, because the sign of α is arbitrary, while that of β is not. (In fact, we got $\beta = \alpha$ by choosing a convention such that positive α required increased area—so giving positive β.) Furthermore, we know that we can change the power of each z_{ij} by multiples of 2 (say, by $2l$) without changing the exchange parameter α (which is only defined mod 2). Increasing the power by $2l$ changes the rate at which added particles consume area (and hence Hilbert space) in the LLL; that is, this changes β—also by $2l$. Hence we see that one can change α (by a sign convention) without altering β; and one can change β (by multiples of 2) without affecting α.

We can clear all of this up by (somewhat reluctantly) defining one more parameter, which we will call γ. This is the power with which $|\Psi(i, j)|$ vanishes as $|z_{ij}| \to 0$; it also determines the density—since it is γ which gives the fictitious charge of the classical two-dimensional, one-component plasma [18] which is mimicked by $|\Psi|^2$. Hence an amended version of (10) is

$$\beta = \gamma \tag{11}$$

where γ is the power of the Jastrow factor $\prod(z_i - z_j)^\gamma$ in (7). Clearly then $|\gamma|$ must equal $|\alpha|$, or at most differ from it by an even integer. Although it is annoying to invent yet another parameter, we feel that γ is needed in this case since α, whose usage and ambiguity are already well established, does not contain the unambiguous information we need.

The above derivation gave (10) (rather than (11)) because we set γ equal to α in (7). This came from the assumption that the 'statistical' flux opposed the external flux. Now, since vortices see electrons as fictitious (external) flux, and hence see (quasi) holes as opposing (statistical) flux, the above argument applies to quasiholes, with $\alpha = 1/m$: $\gamma_{\mathrm{QH}} = \beta_{\mathrm{QH}} = 1/m > 0$. Quasielectrons, in the same coordinate system, should then have the prefactor of Ψ in (7) modified to [5]

$$\prod_{i<j}(z_i^* - z_j^*)^{-\alpha}\,. \tag{12}$$

Here $\alpha_{QE} = \alpha_{QH} = 1/m$ (since the minus sign in the exponent is compensated by the complex conjugation) as asserted above; but (apparently) $\gamma_{QE} = -\alpha_{QE} < 0$. (Intuitively, we can recall that quasielectrons pull electrons in, hence requiring decreased angular momentum and decreased area—that is, negative γ.) This prescription raises the delicate problem [13, 14, 23, 28] of pseudo-wavefunctions for the QE which diverge (rather than vanish) as $z_{ij} \to 0$; however, we will find in section 3.3 that the issue is circumvented by QE built from true, interacting electrons.

We thus see that γ solves the *sign* ambiguity problem associated with α. We also claimed that it resolved the ambiguity of α with respect to adding multiples of two; and it does. However, this means that particles with the same α can have β values which differ by $2l$. Actually, a better way to say this is that *the value of β can depend on the energy scale examined.* In fact, we do not have a finite Hilbert space (D_n in (4)) unless we impose an energy cut-off—which makes good sense for condensed-matter problems, involving low-energy physics in confined systems [1]. It is thus not surprising that β may depend on the cut-off chosen.

To be explicit, for ideal α-anyons with $\alpha > 0$, the total degenerate ground-state manifold has a dimensionality given by (9) which leads to an overall $\beta = \alpha$. But for interacting (non-ideal) anyons, this ground-state degeneracy will be split by the interaction energy. In analogy with the case of electrons in the lowest Landau level [24], one can define pseudopotential parameters $\tilde{V}_{2l}$ which give the energy of a pair of particles in a state of relative angular momentum (RAM) $2l+\alpha$. Then if the $\tilde{V}_{2l}$ differ sufficiently in magnitude, it will be possible to choose different energy scales (corresponding to states in which particles avoid ever-higher RAM). For each such scale there is an appropriate (and unique) value of β. Something entirely similar occurs for quasiparticles in the FQHE. They, too, are interacting, and, as we will show below, one can define multiple energy scales—one for each value of $2l$—corresponding to effective pseudopotential parameters, each with a unique β. In some sense this was (at least part of) Laughlin's insight [25] into the $\nu = 1/m$ states, and is the essence of Jain's [26, 27] 'composite-fermion' approach: at the energy scale of the $\nu = 1/m$ state (which is the Coulomb pseudopotential V_{m-2} for $m = 3, 5, \ldots$) the electrons are fermions with† $\beta = \gamma = m$ (and $\alpha = 1$).

These arguments thus give the same β values for the FQHE quasiparticles as Haldane's—except for some uncertainty about adding multiples of two. We have also connected β with α—at least for the LLL problem. Hence, *assuming* that wavefunctions of the form (7) make sense [5, 18] for quasiparticles, we have that

$$\text{fractional } \alpha \Longleftrightarrow \text{fractional } \beta \tag{13}$$

for quasiparticles in the FQHE.

3.3. β from exact electronic spectra

The expression (13) is particularly interesting because, unlike α (which has been computed from variational wavefunctions [6]), β can be computed independently of *any* variational assumptions, simply by counting states. Furthermore, computational technology has advanced to the point that exact spectra can be computed for finite numbers of interacting electrons over a range of filling fractions including $\nu = \frac{1}{3}$ and the first two daughter states [5, 22] on either side ($\frac{2}{7}$ and $\frac{2}{5}$). Several groups [2, 3, 27, 28, 29] have examined such

† For example, with V_1 as our energy cut-off, we can add $\beta = 3$ electrons to a given area at fixed field until $D_n = 1$, i.e. the Hilbert space is 'full' at this energy scale. This is of course the $\nu = \frac{1}{3}$ state. Further addition of electrons requires an energy V_1 and hence a larger cut-off.

spectra for electrons on a sphere. All these groups obtain the same result for the number of states in the low-energy Hilbert space. For n QH the dimension of the low-energy subspace is, for N_e electrons,

$$D_n^{QH} = \binom{N_e + n}{n}. \tag{14}$$

The corresponding result for n QE is

$$D_n^{QE} = \binom{N_e + 2 - n}{n}. \tag{15}$$

It remains simply to convert these expressions to results at fixed boundary conditions, i.e. fixed total (true) flux (N_Φ flux quanta) through the sphere. This requires writing N_e (which varies with varying n, at fixed N_Φ) in terms of n and N_Φ. For the QH case, N_Φ may be written as $N_\Phi = 3N_e + C + n$, or $N_e = (N_\Phi - C - n)/3$ (where C is a finite-size correction on the sphere [2, 30]). Thus

$$D_n^{QH} = \binom{d_1 + (1 - 1/3)(n-1)}{n} \tag{16}$$

where $d_1 = (N_\Phi + C)/3 + (1 - \frac{1}{3}) = N_e\big|_{n=1} + 1$, i.e. the number of states available to a single charged particle on a sphere which is penetrated by N_e flux quanta—consistent with the claim that quasiparticles see electrons as flux.

Hence the numerical spectra tell us that

$$\beta_{QH} = \tfrac{1}{3} \tag{17}$$

in agreement with (5) above.

Next we want β_{QE}. Since n now represents QE we have $N_\Phi = 3N_e + C - n$ or $N_e = (N_\Phi - C)/3 + n/3$, giving

$$D_n^{QE} = \binom{d_1 + \left(1 - \frac{5}{3}\right)(n-1)}{n} \tag{18}$$

so that

$$\beta_{QE} = +\tfrac{5}{3}. \tag{19}$$

(Again $d_1 = N_e\big|_{n=1} + 1$.)

In other words, quasielectrons built out of real interacting electrons reject $\beta = -\frac{1}{3}$. *This is not surprising since* $\gamma = -\frac{1}{3}$ *puts the* QE *on top of one another, incurring an* unacceptably large Coulomb penalty [28]. The interesting property of quasielectrons is that, regardless of the value of $2l$ chosen, their β parameter cannot fall into the naively expected range $0 \leqslant \beta \leqslant 1$. Haldane's original argument gives them a *negative* β, which would cause the many-particle Hilbert space to grow with increasing n even *faster* than that of ideal bosons—by causing the 'number of states' d_n available to the nth particle to actually *increase* with n. Instead, we find that the QE 'fill states' even more rapidly than do ideal fermions.

3.4. Are α and β ambiguous?

One commonly hears the claim that 'statistics are ambiguous in two dimensions'. A better translation of this statement is, 'one cannot deduce α statistics from a wavefunction's symmetry'. This is because particles of any α statistics can be represented by either fermions or bosons with attached fictitious flux. This 'ambiguity' allows one, for instance, to represent fermions with symmetric (bose) wavefunctions [31, 32]. The particles are nevertheless fermions due to the flux attached to the 'bosons'; this may be ascertained using (singular-) gauge-invariant† measures such as a Berry's phase [33].

Similarly one could argue that state counting does not give an unambiguous measure of statistics. For instance, the result (14) can be explained from two completely different viewpoints [1, 3]. First, suppose for the moment that QH can be treated as bosons which see the background electrons as N_e quanta of effective flux. Thus the low-energy states are those of n bosons moving in a lowest Landau level of dimension $N_L^{\text{eff}} = N_e + 1$. This subspace has dimension $\binom{N_L^{\text{eff}}+n-1}{n}$, which is exactly (14). This viewpoint underlay Haldane's original 'bosonic' approach to the hierarchy [22].

The second way to explain the dimension (14) is based on the 'integer-mapping' approach used by Jain and co-workers [26, 27]. Each n-QH state near $\nu = \frac{1}{3}$ can be mapped [34] to an n-hole state near $\nu = 1$ by multiplying the latter by a Jastrow factor (on a disk, by $\prod(z_i - z_j)^2$). This mapping is one-to-one and, for the purposes of state counting, exact. (All the states below the kinetic-energy gap near $\nu = 1$ can be written as the filled Landau level wavefunction (which contains a factor $\prod(z_i - z_j)$) times symmetric polynomials $Q(z_1, z_2, \ldots, z_{N_e})$. All the states below the interaction (V_1) gap near $\nu = \frac{1}{3}$ can be written as the Laughlin wavefunction (which contains a factor $\prod(z_i - z_j)^3$) times the same polynomials Q.) The n-hole system has N_e electrons in $N_e + n$ quanta; there are $\binom{N_e+n}{n}$ of these, again in agreement with (14). Thus the dimension of the subspace lying below the interaction gap can be obtained by arguments treating the QH as bosons (as in the bosonic hierarchy approach), or as fermions (as in the integer-mapping approach).

A similar statement holds for (15)—with the proviso that 'bosons' be broadly defined to include $\beta = 2$. Hence one might argue that β-statistics are ambiguous in the FQHE.

We view Haldane's approach to β as the analogue of the gauge-invariant approach to α. The virtue of Haldane's approach is that it requires d_1 (the 'effective Fock-space dimension') to remain *fixed* while n is varied, at fixed boundary conditions [1]. Once an energy scale (and hence D_n) is identified, this prescription gives a *unique* result for β. At the scale of the interaction gap in the FQHE, the corresponding β values are given by (17) and (19) above. These values are of course fractional; and they rely on no variational assumption (beyond the minimal assumption that we can count the quasiparticles). Furthermore, assuming that one can use an effective Hamiltonian and effective wavefunctions to describe the quasiparticle degrees of freedom, one gets from (13) an independent confirmation of fractional α.

3.5. *Pruning the hierarchy, using β*

We can easily identify first-generation 'daughter' states of $\nu = \frac{1}{3}$. They occur when the low-energy space consists of a single, non-degenerate ground state, separated from the excited states by a gap which measures the strength of the interaction between quasiparticles. In our present formulation, this means setting $D_n = 1$ in (16) or (18), and solving for n. Converting the pair (n, N_Φ) to (N_e, N_Φ) then gives the usual assignments [30] for the daughter states—for instance, the QH daughter $\frac{2}{7}$ and the QE daughter $\frac{2}{5}$, arising from the parent $\frac{1}{3}$. We note that there is no sensible solution to $D_n = 1$ (where 'sensible' means that d_1 and n are both large and positive, while $|\beta|$ is of order 1) that involves a negative

† A singular gauge transformation amounts to exchanging true (but confined and impenetrable) flux by modified, and in general multivalued, boundary conditions (and vice versa). For instance, fermions with attached point flux and an antisymmetric wavefunction (and in 2D) are, after a singular gauge transformation, α-anyons with multivalued wavefunctions. The exchange phase η however is invariant under the transformation, being dynamically generated in the former representation and built into the wavefunction in the latter. See [7, 23, 31].

β. Hence, the assignment $\beta_{QE} = -1/m$ is inconsistent with the standard hierarchy theory [22].

We also note (as noted previously [28, 35]) that the gap which stabilizes the $\frac{2}{7}$ state is of order V_3, while that for $\frac{2}{5}$ is of order V_1. This occurs because the former state (and some of the excited states at $\nu = \frac{2}{7}$) are composed entirely of electronic states which avoid relative angular momentum (RAM) 1 between pairs; hence the $\frac{2}{7}$ ground state is distinguished from its low-lying excited states by an energy of order V_3. At $\nu = \frac{1}{3}$, the ground state is the unique state avoiding V_1; hence the gap is of $\mathcal{O}(V_1)$. Finally, for $\nu > \frac{1}{3}$, all states include some electron pairs with RAM 1, such that, in general, the interactions between the QE [35] are of $\mathcal{O}(V_1)$, as is the gap at $\nu = \frac{2}{5}$. These statements may be verified by varying the $\{V_l\}$ numerically, and observing the resulting spectra.

We have not yet penetrated more than one layer deep into the hierarchy of states using the above approach of studying exact spectra for small systems of electrons. However, we can learn something about alternate daughters at the same level. We view the $\frac{2}{7}$ state as the unique state in which the QH avoid RAM $\frac{1}{3}$. In analogy with the electronic pseudopotentials we define $\tilde{V}_{2l}$ as the interaction energy of two QH in RAM $\left(2l + \frac{1}{3}\right)$. Then the $\frac{2}{7}$ state is the unique state avoiding $\tilde{V}_0$ at that filling, and its gap is $\mathcal{O}(\tilde{V}_0) \sim \mathcal{O}(V_3)$. One can then imagine a unique ground state at a lower density of QH (higher electron density) at which all the QH avoid RAM $2+\frac{1}{3}$, and so avoid the energy cost $\tilde{V}_2$. This is a state in which all the QE have 4 extra zeroes and so experience $\tilde{V}_4$ while avoiding $\tilde{V}_2$ (in analogy with electrons at $\nu = \frac{1}{5}$ feeling V_5 while avoiding V_3).

For electrons the $\{V_l\}$ are monotonic in l: $V_1 > V_3 > V_5 \ldots$. Hence the state at $\nu = \frac{1}{5}$ in which electrons participate in RAM 5 while avoiding 3 is not only unique; it is also the ground state. The analogy with quasiparticles and the $\{\tilde{V}_{2l}\}$ will then hold if this set is also monotonic in $2l$. The $\{\tilde{V}_{2l}\}$ have been estimated from exact numerical few-electron spectra by us [2, 4] for QH and for QE, and by a more elaborate procedure using trial wavefunctions (for QE) by Béran and Morf†. Our own approach relied on the fact that, for $n = 2$, subsets of states with various values of $2l$ can be identified‡ with multiplets of total angular momentum in the electronic spectrum, using (4) with $n = 2$ and $\beta = 2l + \frac{1}{3}$.

Our analysis, as well as that of Béran and Morf [36], gives the result that the set $\{\tilde{V}_{2l}\}$ is *not* monotonic: in every case, we find $\tilde{V}_0 > \tilde{V}_4 > \tilde{V}_2$. These results imply that the hypothesized state in which all the QH feel $\tilde{V}_4$ while avoiding $\tilde{V}_2$ is not the ground state at that filling. This means that the analogy with the electronic 'horizontal' hierarchy of increasing β values breaks down at this filling—which is, in electronic units, $\nu = \frac{4}{13}$ (QH) and $\nu = \frac{4}{11}$ (QE).

Hence we find that, according to standard hierarchy theory, these filling fractions should not be stable incompressible states for polarized electrons. This conclusion was also offered as a conjecture for the $\frac{4}{11}$ state by Gros and MacDonald, who studied trends in the chemical potential jumps as a function of filling for finite systems of electrons [30]. Both of these

† The set $\{\tilde{V}_{2l}\}$ have been estimated for QE, using a procedure involving variational wavefunctions [36]. They found $\tilde{V}_0 > \tilde{V}_4 > \tilde{V}_6 > \tilde{V}_2$, i.e. qualitatively similar results to ours.

‡ For example, $D_n = 28$ for $N_e = 6$ and $n = 2$ with $\beta = \frac{1}{3}$ (quasiholes). These states can be clearly identified in the spectrum. The 28 states consist of multiplets of total angular momentum $L = 0, 2, 4, 6$ and hence dimension $1, 5, 9, 13$. Setting $\beta = 2l + \frac{1}{3}$ with $2l = 0, 2, 4, 6$ then gives $D_n(2l) = 28 = 13 \oplus 9 \oplus 5 \oplus 1$; $15 = 9 \oplus 5 \oplus 1$; $6 = 5 \oplus 1$; and 1, respectively. Thus we identify $2l = 6$ with the singlet; $2l = 4$ with the $L = 2$ multiplet; and so on. We then take $\tilde{V}_{2l} = (E_{2l} - E_0) - 2(E_1 - E_0)$, where E_{2l} is the energy of the relevant 2-QH multiplet and $E_{0,1}$ is the lowest-energy state with 0 or 1 QH present.

conclusions are apparently consistent with experiment† [38].

3.6. *Summary: α and β in the* FQHE

Thus, finally, a partial answer to 'where are the anyons?' may be had from the above: the quasiparticles in the FQHE possess both fractional α and fractional β. We believe that these are unambiguous results from *theory*. Experimental verification remains to be found. There is evidence for the fractional *charge* of the quasiparticles [39] (and some controversy [40] about it), and a suggestion [41] for a modification of these experiments which might test for fractional α.

We also offer a small increment to the small existing body of testable predictions from anyon models, namely, the instability of the $\frac{4}{11}$ and $\frac{4}{13}$ states. This amounts to cutting branches off the standard hierarchy 'tree' of states [8]; this tree itself is the main experimental prediction of the standard hierarchy theory, which models the quasiparticles as anyons.

4. High-temperature superconductors

We now turn to the question of fractional statistics in the HTSC. The suggestion that quasiparticles in the HTSC may not have the quantum numbers appropriate to free electrons was first made by Anderson [17]. Subsequently, Laughlin [11, 20] argued that the spin excitations ('spinons') of a two-dimensional quantum spin-$\frac{1}{2}$ antiferromagnet should have fractional statistics with $\theta = \alpha\pi = \pi/2$, by mapping the spin problem to that of hard-core bosons in a magnetic field, at $\nu = \frac{1}{2}$. Laughlin further argued that the charged excitations of the doped magnet should take on the same α-statistics (exchange phase) as the spinons, and that such an exchange phase could by itself give rise to long-range phase coherence at low temperatures—thus giving birth to the notion of 'anyon superconductivity'.

A consequence of Laughlin's idea was immediately pointed out by Kivelson and Rokhsar [42]: (α-) fractional statistics breaks time-reversal ($\mathcal{T}$) symmetry, since $e^{i\theta} \xrightarrow{\mathcal{T}} e^{-i\theta}$. In the case of the FQHE, the broken $\mathcal{T}$ symmetry is an obvious consequence of the externally-applied magnetic field; however in the case of the HTSC the symmetry must be broken spontaneously (as noted by Laughlin [42]).

Subsequent attempts to test or extend Laughlin's idea may be grouped into two classes, addressing the following two questions. (i) Is there a good 'derivation', starting from a physically reasonable microscopic Hamiltonian with unbroken symmetry, of the fractional statistics of the excitations in the HTSC? (ii) What are the promising experimental tests of anyon superconductivity? (For the remainder of this section we take 'anyons' and 'anyon superconductivity' to mean explicitly α-anyons, as is customary in this area. β-anyons will be considered again only in the concluding section.)

An example of (i) is Laughlin's aforementioned mapping of the spin problem to hard-core bosons at $\nu = \frac{1}{2}$. This amounts to a variational argument, as do most other arguments addressing this question [19]. We feel that, in spite of considerable work on the problem, there is no truly compelling theoretical argument for fractional α in doped antiferromagnets. Hence we will not address this work here.

The idea of anyon superconductivity has attracted considerable attention in spite of the relative weakness of this chain in the logic, in part because of the beauty and novelty of

† There has been evidence of FQHE behaviour at $\nu = \frac{4}{11}$; but it appears to be a state of mixed spin polarization. See [36, 37].

the idea. The next link in the chain—that fractional α can give rise to a novel form of superfluid—appears to be stronger. Again, however, we will not review this link as it has been reviewed elsewhere; see, for example, [7] and [10].

We then arrive again at the question 'Where are the (α-) anyons?', which leads naturally to (ii) above: how do we detect them? There has been considerable effort expended on question (ii). Some of the early examples include [12, 43–45]. This work has given rise to several suggestions for experimental tests of anyon superconductivity. All of the tests but one have given an uncontroversial null result, and so will not be mentioned further here†. Instead we will concentrate (section 4.2) on giving a detailed history and analysis of experiments on optical rotation in the HTSC, seeking signs of broken $\mathcal{T}$ symmetry. Here we find the experiments have generated considerable controversy (which has somewhat abated), while the theory (as is typical) has provided only a qualitative guide as to what is to be expected.

There is another group of experiments, seeking spontaneous local magnetic fields in the HTSC, which also will be briefly discussed (section 4.3). Here we find no controversy from experiment (null results), while theory fails to give an uncontroversial estimate for the magnitude of the expected effect.

4.1. The problem of many planes

There is one feature of fractional (α) statistics which is uncontroversial, namely, the broken $\mathcal{T}$ symmetry. Even here there is some difficulty, however, since (unlike the FQHE) the HTSC consist of many stacked planes in which $\mathcal{T}$ is broken spontaneously (if at all). It is then necessary to assume some kind of ordering (including, in principle, no order) of the broken symmetry, in order to even estimate the magnitude of the possible effects to be seen in an experiment.

There are reasons both from experiment and from theory to favour the hypothesis of 'antiferromagnetic' ($\cdots + - + - + \cdots$, or 'AFM') order of the planes, over a uniform 'ferromagnetic' ($\cdots +++++ \cdots$, or 'FM') ordering of the broken symmetry. For one thing, it seems difficult to reconcile the various null experimental results with a FM scheme, which would likely [43] have gross macroscopic consequences, such as an anomalous magnetic susceptibility, Faraday rotation, etc.

A second reason comes from theory. Rojo and collaborators [47] have shown rather convincingly that, in the limit that tunnelling between planes is small (which is a typical assumption for the anyon model), there is an energetic preference for AFM order of the broken symmetry. This conclusion was obtained (1) from an exact solution of a one-dimensional model involving particles confined to coaxial rings in current-carrying states, with the particles coupled (between rings) by an ordinary scalar potential $V(\theta_{ij})$; (2) from exact numerical results for small numbers of anyons on planes, with again a scalar interplane potential; and (3) from a general analysis, to second order in the coupling, of $\mathcal{T}$-violating systems coupled with a scalar potential. The energetic difference between FM and AFM alignment vanishes at first order in the coupling, and hence is in general insensitive to the sign of the scalar potential (attractive or repulsive). One conclusion to be drawn from this work is that there is a non-classical electrostatic coupling between circulating currents which favours (energetically) *opposing* circulation of nearby current elements. Thus, for instance, this effect would cause coaxial rings of current, at fixed magnitude of the current, to favour opposing or 'AFM' order of the currents. Similarly, planar domains of α-anyons

† An example is an experiment seeking an offset in the flux quantization spectrum of a multiply-connected HTSC sample by [46].

would have (assuming a finite net moment from circulating currents around the domain edge—see section 4.3) an energetic preference for opposing currents, arising purely from a non-$\mathcal{T}$-violating coupling such as the Coulomb interaction.

This effect thus competes with the classical (magnetostatic) effect, which, of course, favours FM coupling of currents circulating in coaxial rings or in stacked planes. It also competes with the coupling due to interplane tunnelling [48]; however, order-of-magnitude estimates for the three effects [47] suggest that the tendency to AFM order will predominate in the HTSC.

4.2. The closest approach of theory and experiment: AFM ordering of planes versus optical rotation

Although the above is certainly not conclusive, AFM ordering of the planes has been of some interest for a third reason. A series of experiments [56–60] has been performed on the HTSC, seeking evidence for broken $\mathcal{T}$ symmetry in the response of these materials to circularly polarized light—a possibility first suggested by Wen and Zee [44]. These experiments have attracted considerable interest since they offer—though not unambiguously—some evidence for broken $\mathcal{T}$. Furthermore (as we will argue below—following work by Dzyaloshinskii [54] and by Canright and Rojo [55]), the most promising hypothesis involving broken $\mathcal{T}$ assumes AFM order of the planes.

In this subsection we will review this series of experiments and their analysis. In our view the apparent viability of the hypothesis of broken $\mathcal{T}$ in the HTSC has gone through some large changes (up and down) in the course of the last three years. Hence it is of interest to trace the experiments in historical order.

First we establish some notation. We consider light which is incident normal to the planes (consistent with the experiments), and resolve the light in a basis of circularly polarized (CP) waves: + and −†. Reflection and transmission coefficients are R and T respectively. Since (3D) spatial inversion $\mathcal{P}$ will also be important we will distinguish coefficients for incidence on the 'right' (r) from those for incidence on the 'left' (ℓ). Thus, for example, for unit amplitude +-CP light incident on the left, R^{ℓ}_{++} and T^{ℓ}_{++} give the amplitude of +-CP light reflected on the left, and transmitted to the right, respectively (for details see [55]). Finally we adopt a common notation for the HTSC materials: 123 for $YBa_2Cu_3O_{7-x}$, 2212 for $Bi_2Sr_2CaCu_2O_8$, and 214 for $La_{2-x}Sr_xCuO_4$.

The first experiments, by Lyons *et al* [49] measured a signal proportional to $S_{L_1} \equiv (|R_{++}|^2 - |R_{--}|^2) + \delta$. The term in parentheses is zero in the absence of broken $\mathcal{T}$, as shown by Halperin [56]. The term δ represents another, contaminating signal in the experiment (not necessarily smaller than the first) which might arise from broken $\mathcal{P}$ and unbroken $\mathcal{T}$. Lyons *et al* found a non-zero value for S_{L_1} in thin films of 123 and in bulk crystals of 123 and 2212, but due to the presence of the term δ could not definitely conclude that broken $\mathcal{T}$ had been observed. There were, however, several checks against artifacts: no signal was observed for a polished silicon sample, nor for a gold film on the substrate. Also, and most interestingly, a 123 sample which gave a non-zero signal was then annealed in nitrogen to render it insulating; subsequently the insulating sample was found to give zero signal. Finally, Lyons *et al* reported a temperature dependence of their signal which, they argued, could not reasonably be ascribed to any parity-breaking transition which might give rise to a non-zero δ.

† These symbols refer to *absolute*—i.e. independent of the sign of the wavevector k—circular polarizations, rather than to chirality (right- or left-handedness with respect to k). Hence time reversal takes $+ \rightleftharpoons -$ (while leaving chirality unaffected).

Soon after these results were announced, Spielman *et al* [50] reported measurements of $S_{S_1} \equiv \arg(T^{\ell}_{++}/T^{r}_{--})$ on thin films of 123. This group used a fiber-optic loop allowing for the interference of two counterpropagating beams of light; hence the experimental technique almost perfectly reproduced the idealized gedanken experiment which compares an experiment (T^{ℓ}_{++}) and its time-reverse (T^{r}_{--}). These experiments gave a null result with very high sensitivity.

Subsequently, Weber *et al* [51] measured a signal similar to that of Lyons *et al* in reflection, for single-crystal 123. The same group also measured the rotation in transmission of linearly polarized light, aligned on input with a principal axis of the (orthorhombic) sample (thin-film single crystals of 2212). This latter signal is proportional to

$$S_{W_1} \equiv \arg\left(\frac{T^{\ell}_{++} + T^{\ell}_{+-}}{T^{\ell}_{-+} + T^{\ell}_{--}}\right). \tag{20}$$

Weber *et al* found non-zero (even, in some cases, rather large) values for each of these signals, with a temperature dependence that was qualitatively like that found by Lyons *et al* [49]. Note, however, that neither of these measurements is sensitive only to non-reciprocal† effects; that is, both can be non-zero even if $\mathcal{T}$ is a good symmetry for the sample. As evidence for the role of broken $\mathcal{T}$, Weber *et al* made ten field-cooling trials (turning off the field above T_c, and measuring below T_c)—three in reflection and seven in transmission. In every case they found consistency between the sign of the measured signal and the sign of the cooling field [51]. Making the common assumption that the broken-symmetry order parameter might form domains in each plane whose size is limited by the twin domain size of the samples, they conjectured that the large magnitude of their observed signal could be ascribed to the large twin domain size in their samples.

These early experiments gave an extremely unclear picture. As noted by Spielman *et al* [50], and later verified by a detailed analysis [55], the signal S_{S_1} measured by Spielman *et al* is the only purely non-reciprocal signal in the above experiments—that is, it is the only signal that conclusively implies broken $\mathcal{T}$ if it is non-zero. However, it can also be zero if $\mathcal{T}$ is broken. This possibility was suggested by Dzyaloshinskii [54], who examined the implications for these experiments of the AFM model for the HTSC. Dzyaloshinskii observed that the AFM model has interesting symmetry properties which depend on the number of planes per chemical unit cell of the compound. In particular, materials with one plane per unit cell (such as 214) have the symmetry of a one-dimensional Ising antiferromagnet—that is, except for surface effects, the material is invariant under $\mathcal{T}$. In contrast, for two planes/unit cell (123 and 2212) the bulk AFM does not come back to itself under $\mathcal{T}$, but is invariant under $\mathcal{PT}$ (where, again, $\mathcal{P}$ is 3D spatial inversion), since $\mathcal{P}$ switches the two planes, and $\mathcal{T}$ inverts the order parameters. Dzyaloshinskii noted that $\mathcal{PT}$ invariance‡§ ensures that $T^{i}_{++} = T^{i}_{--}$ ($i = r, \ell$), but does not constrain the corresponding reflection coefficients. Hence his analysis suggested a possible reconciliation of most of the above

† A non-reciprocal signal S_{NR} has the property $\{S_{NR} \neq 0\} \Longrightarrow \{\text{broken } \mathcal{T}\}$, or the converse $\{\text{unbroken } \mathcal{T}\} \Longrightarrow \{S_{NR} = 0\}$. Hence $\{S_{NR} = 0\}$ does not tell us anything definite about broken $\mathcal{T}$.

‡ Time reversal takes $T^{\ell,r}_{++}$ to $T^{r,\ell}_{--}$; $\mathcal{P}$ then takes the latter to $T^{\ell,r}_{--}$ so we get $T^{\ell,r}_{++} = T^{\ell,r}_{--}$ for $\mathcal{PT}$-invariant materials. In contrast $\mathcal{PT}$ takes $R^{\ell,r}_{++}$ to $R^{r,\ell}_{--}$ so that $\mathcal{PT}$ symmetry does not force $R^{\ell,r}_{++} = R^{\ell,r}_{--}$. Note that the convention here is that + and − are *absolute* circular polarizations, i.e. defined with respect to time rather than relative to the $\hat{k}$-vector of the light. See [55] for details.

§ Dzyaloshinskii [54] also made the stronger statement that $\mathcal{PT}$ symmetry forbids optical rotation in transmission—which would make the signal S_{W_1} (20) of Weber *et al* [51] zero for $\mathcal{PT}$ symmetry. However one can show that this statement is true only in the case of rotational symmetry about the propagation axis—which makes the 'off-diagonal' (T_{+-}, T_{-+}) terms zero in (refsw2).

results, as well as an interesting test, namely a comparison of the behaviour of 214 with that of the two-plane materials.

This test was performed by Lyons *et al* [57], who used an improved apparatus to cancel the unwanted signal δ, thus measuring $S_{L_2} \equiv (|R_{++}|^2 - |R_{--}|^2)$ on 123 films and on a single sample of 214. The interesting result is that the 214 sample gave no signal; however, some of the 123 samples also gave no signal—hence, unfortunately, a consistent difference between 214 and 123 can be neither conclusively confirmed nor ruled out. Also, although S_{L_2} is purely non-reciprocal so that a non-zero signal implies broken $\mathcal{T}$, the puzzling sample dependence of the results somewhat weakens the implication.

Canright and Rojo [55] carried out a detailed symmetry analysis of the above experiments. They confirmed that $S_{S_1} = \arg(T^{\ell}_{++}/T^{r}_{--})$ is strictly zero if the sample obeys ($\mathcal{PT}$ + orthorhombic) symmetry. They also showed that the transmission results of Weber *et al* [51] are allowed by these same symmetries, due to an interplay between the lack of rotational invariance and the broken $\mathcal{T}$. Hence, on symmetry grounds alone, Canright and Rojo found that all the reported experiments could be qualitatively reconciled. They also noted that early work [58] with 'magnetoelectric' (i.e. $\mathcal{PT}$-invariant) antiferromagnets had shown that the order parameter of such materials could be reliably biased by cooling in a uniform magnetic field. Hence even the field-cooling results of Weber *et al* [51] might be reconciled with the AFM model.

The apparent clarification offered by this analysis was short-lived, however. Spielman *et al* [52] modified their apparatus to test the AFM hypothesis, by measuring a non-reciprocal signal in reflection ($S_{S_2} \equiv \arg(R^{i}_{++}/R^{i}_{--})$)—a signal which is not identically zero given $\mathcal{PT}$ invariance—in 2212 crystals and in 123 films. This experiment thus measured a quantity very similar to that measured by S_{L_2}, and which is expected to be of roughly the same magnitude. Spielman *et al* again got a null result to high precision.

Subsequently, Lawrence *et al* [53] attempted to closely replicate the experiment of Lyons *et al*, thus also measuring S_{L_2}, on single crystals and films of 123. They obtained a null result for all samples tested, after discovering and eliminating artifactual effects. In particular, they showed that surface roughness, coupled to apparatus imperfections—that is, effects which have nothing to do with $\mathcal{T}$ violation—could give a sizeable signal even for apparatus which ideally is only sensitive to non-reciprocal effects. They pointed out that temperature dependence of the kind seen by Lyons *et al* may be caused by condensation on the sample of contaminants such as ice and air, if the vacuum is not sufficiently good.

These experiments thus directly challenge the results of Lyons *et al*, which are the strongest positive results implying broken $\mathcal{T}$ symmetry in the HTSC. Further weight in favour of the null hypothesis was offered by Shelankov [55, 59], who showed that the substrate (required for the 123 samples) in the transmission experiments of Spielman *et al* [50] invalidated the symmetry assumptions of Dzyaloshinskii [54] and of Canright and Rojo [55], such that the transmission experiments with substrate should have given a signal, possibly comparable to that seen in reflection. Shelankov also argued [55] that surface effects in an AFM, one-plane/unit cell material such as 214 should be as large as the nominally bulk effects in the two-plane materials 123 and 2212. Subsequent calculations for finite stacks of $\mathcal{T}$-breaking planes by Canright and Rojo [55] tended to support this argument; however, the calculations could not be extended to the true bulk limit.

The history of these experiments is thus rather tortuous. A conservative evaluation of the results described would conclude that $\mathcal{T}$ is not broken in the HTSC. There is in fact no clear and unchallenged evidence for broken $\mathcal{T}$ symmetry in the HTSC from these experiments, and the majority of the experiments are consistent with the null hypothesis that $\mathcal{T}$ is unbroken in these materials. There are numerous technical details which may however undermine these simple observations; in particular, Lyons [57] has emphasized that differences in laser spot profile may account for the differences between his own results and those of Spielman and of Lawrence†. The dependence on cooling field of the Weber experiments [51] is also puzzling.

We note that all of the above experiments have been analysed in yes/no terms, i.e. only qualitatively. There are some estimates of the optical properties of one or many layers of α-anyons [44, 60]; however these estimates are subject to significant uncertainty. In any case the experimental results discussed above fail to support the hypothesis of anyon superconductivity already at the qualitative level.

4.3. Spontaneous local fields from α-anyons

In this subsection we briefly review the search for local magnetic fields in the HTSC. The suggestion of a spontaneous, *orbital* magnetic moment from α-anyons was first made by Halperin *et al* [43]. These authors considered a model involving two 'species' of α-anyons, arguing that these species arise as a result of the broken symmetry (which doubles the unit cell in real space). For simplicity (and for easy comparison with other results) we will discuss results for a single species; also, unless otherwise stated, a single plane of ideal anyons with $\alpha = \frac{1}{2}$ will be assumed. For these values Halperin *et al* found (here we neglect the overall sign) $M_0 = \left(\frac{3}{4}\right)\mu_B$ at $T = 0$ where M_0 is the spontaneous orbital magnetic moment per particle in units of the effective Bohr magneton $\mu_B = e\hbar/2m^*c$. (The effective mass is of order 1–10 m_e for the carriers in the HTSC.)

Halperin *et al* obtained this value by adiabatic continuation from the mean-field state (where the 'attached flux' giving rise to α is treated at the mean-field level), and argued that it is exact. Subsequent experimental attempts [61] to measure the resulting magnetic field in HTSC samples failed to find any, with a sensitivity threshold at least an order of magnitude below the estimate of Halperin *et al.*

These developments, although discouraging for the anyon hypothesis, at least offered an apparent, clean and quantitative disagreement between theory and experiment. Subsequent theoretical efforts have only muddied the issue: Kitazawa [60], also using arguments based on mean-field theory, found a temperature-dependent M_0 which is zero in the superconducting state and of order $\lesssim 0.1\mu_B$ in the normal state. This picture was supported by exact zero-temperature numerical results for a few anyons by us [62] which also gave $M_0(T = 0) = 0$; by our use [62] of the virial expansion (which necessarily describes the normal or high-temperature state, but avoids the mean-field approximation), which gave $M_0 \lesssim \left(\frac{1}{8}\right)\mu_B$ and by the exact (few-body) finite-temperature calculations of Canright and Rojo [63], which interpolated nicely between these results. Subsequent calculations by Halperin and Gelfand [64], again using mean-field theory but including the Coulomb interaction, gave a substantial zero-temperature M_0, in line with the original non-interacting result of Halperin *et al* [43]. Finally, we mention the virial-expansion calculation of Yi and Canright [65], which included a long-range repulsion varying as $1/r^2$ (which allowed an exact solution); this calculation gave a reduction of the normal-state M_0 by orders of magnitude from the non-interacting case.

We thus see that theorists cannot agree on M_0 for ideal or interacting anyons, at low or high temperatures. While we believe that the picture ($M_0 = 0$ in the superconducting state, M_0 = finite in the normal state) is probably correct for ideal α-anyons, there is not

† For example, a magnetic garnet sample, with domain size approximately that of the HTSC twins, was tested in all three laboratories. The signal obtained in the apparatus of Spielman and of Lawrence, was significantly reduced relative to that seen by Lyons ([57] and K B Lyons, private communication).

general agreement on this conclusion; and the effects of interactions are not clear. Hence we feel that there is no firm theoretical basis for the experiments to test; the estimates range everywhere from zero to much bigger than the experimental threshold of sensitivity.

4.4. Where are the anyons in the HTSC?

Let us summarize the above. Such a summary can be quite brief: (i) there are no confirmed experimental results which support the hypothesis of α-anyons (or even broken $\mathcal{T}$) in the HTSC; (ii) there are no firm quantitative predictions† from theory for experimentalists to test. Point (ii) should not surprise us, since the state of anyon theory is primitive, and the problem is difficult. Points (i) and (ii) together, however, leave little room for enthusiasm for the anyon-superconductivity model for the HTSC.

5. Summary, conclusions, and prospects

We have tried to show in this review that Haldane's fractional β is a concept at least as fundamental and useful as fractional α. Let us assume that we have succeeded, and ask where the idea leads.

For one thing, we note that section 4 makes no mention of β. In fact, it describes the theoretical difficulties and the experimental discouragement of a model (anyon superconductivity) involving fractional α with an apparently ill-defined β. Given the demonstrated applicability of fractional β-statistics in the FQHE, the question then naturally arises: what is the appropriate role and utility (if any) of fractional β in anyon models of doped two-dimensional antiferromagnets?

Laughlin's original proposal [11, 20] of $\alpha = \frac{1}{2}$ for the charge-carrying quasiparticles in the HTSC was based on a strong analogy with the FQHE. The similarity of the variational wavefunctions for the two problems implies that the quasiparticles in each case are vortices. We then recall Haldane and Wu's result [21] that the quantum mechanics of vortices in a 2D fluid is that of (fictitious) charges in the lowest Landau level (LLL). Finally, invoking our arguments from section 3 that (in the LLL) fractional α should always be accompanied by fractional β, we find that α-anyons in Laughlin-like chiral spin liquids should in fact have a fractional β. We then inquire whether such a feature has in fact been incorporated in the theoretical models which have been used to study anyon superconductivity.

In some cases—lattice models [32] with α-anyons—β naively appears to be 1 [1]. At least the total Hilbert space, including all possible many-anyon states, has a dimensionality that gives $\beta = 1$: each particle exhausts one site, which represents one state in the Hilbert space. However, we have shown in the discussion of sections 2 and 3 that β depends on the energy scale being investigated. It is then possible that there exist low-energy sets of states in these lattice calculations which would be described by a fractional β; this interesting possibility remains to be tested. We also know little about β for continuum anyons in two dimensions in the absence of a magnetic field. Semions $\left(\alpha = \frac{1}{2}\right)$ in no field appear, like bosons, to have a superfluid ground state, and to have a low-energy linearly dispersing mode. For finite systems (e.g. α-anyons on a sphere [67]) one can count the number of states in this mode, which lies below an energy gap. We find that this subset can be described, at least in the thermodynamic limit, by an exclusion coefficient $\beta = 0$—so that the semions,

† The order parameter for anyon superconductivity (with $\alpha = \frac{1}{2}$) has recently been predicted to be a complex d-wave without nodes, see [66]. It is not clear, however, that this is a necessary consequence of the assumption $\alpha = \frac{1}{2}$; see also [43].

restricted to this low-energy (superfluid) subspace, are bosons for state-counting purposes [68]. Yet, in this model, the overall (truncated) Hilbert space is built up as if the particles are truly fermions in the sense of $\beta = 1$. Thus we find a mixed and unclear picture from these numerical studies of α-anyons; but, to date, we see no evidence for a fractional β.

Now, although the theoretical justification for anyon superconductivity via doped, chiral spin liquids [19] is not compelling, an essential ingredient appears to be the close analogy with the FQHE. Hence, in such spin liquids, the spinons are vortices, and the LLL picture is correct—so that their β should be $\frac{1}{2}$, as argued by Haldane [1]. Thus, models of anyon superconductivity which ignore the possibility of fractional β (as do essentially all calculations in the field) appear to us to be *inconsistent with the microscopic basis for the model*—which involves, again, vorticity of the excitations, the associated fictitious field [15], and the consequent fractional β. Therefore, even though the theory and consequences of fractional β remain to be explored, we find it difficult to place confidence in models which ignore such a possibility.

Based on our arguments in section 3.2, a set of flux-carrying fermions (or bosons) *in the lowest Landau level* will, as it were, automatically acquire a fractional β from a fractional α. Hence, models of spinons (or holons) with fractional α may be consistent if they include, in an appropriate way, a fictitious magnetic field. To our knowledge, [43] is one of few attempts to do so.

A possibility which remains to be explored is that of fractional β without broken time-reversal symmetry, arising, say, in spin liquids (doped or undoped) which are not chiral. Possibly, such a picture may capture Laughlin's insight [11] that spin-$\frac{1}{2}$ excitations in a disordered, spin-$\frac{1}{2}$ antiferromagnet are in an important sense *fractional*, without the necessity of invoking the chirality and concomitant broken symmetry. We note that Haldane's [1] simple argument giving $\beta = \frac{1}{2}$ for spinons does not involve any broken symmetry; in fact, it relies solely on the assumption that the spinons are well defined. Hence, we believe that the argument for fractional β for spinons is more convincing than those arguments giving fractional α.

While the consequences of fractional α are only partly known, the consequences of fractional β have hardly been addressed. We believe, and have argued above, that both concepts (fractional α and fractional β) are appropriate for the quasiparticles in the FQHE. We have also shown that the notion of fractional β can be *useful*, since it allowed us to extract interesting and non-trivial pseudopotential parameters representing the effective interactions between quasiparticles. These parameters in turn led to a testable hypothesis regarding the (lack of) stability of the FQHE at filling fractions $\nu = \frac{4}{13}$ and $\frac{4}{11}$.

In summary, we find that there are now two workable definitions of fractional statistics. These two ideas are coupled together for particles in the lowest Landau level (13). We find that both ideas are realized in the FQHE. In contrast, we find the case for applying either definition to doped antiferromagnets, and in particular to the HTSC, to be much weaker. Whether these two concepts of fractional statistics will be realized elsewhere in nature, and how useful they will ultimately be, depends on further work, which will, we hope, be guided by further advances in the as-yet primitive theory arising from these ideas.

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